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The wave field generated by a centre of rotation in an unbounded elastic medium with a semi-infinite conical crack☆

N.D. Vaisfel'd, G.Ya. Popov*

Odessa, Ukraine

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ABSTRACT

The dynamic stress intensity factor at the edge of a semi-infinite conical crack when the medium is loaded by a non-stationary centre of rotation is determined. A centre of rotation is understood to be a set of four forces of equal magnitude that act in the same plane and form pairs having the same direction of rotation.[1] If the magnitude of these forces is time dependent, i.e., their application is non-stationary, they form a non-stationary centre of rotation. The solution of the problem required the use of methods of integral transformations and discontinuous solutions, which reduced the problem to an integral differential equation in Laplace transform space. The combined use of the orthogonal polynomial method and time discretization to solve the equation enabled a formula for the stress intensity factor to be obtained.

Non-stationary problems of mechanics for elastic media with cracks are solved by various methods.[2–8] The stress concentration near defects of conical shape in unbound elastic bodies was investigated in earlier studies,[9–11] where these problems were considered in a static formulation. In a non-stationary formulation, the problem has been solved for a conical defect of finite size.[12]

1. Formulation of the problem

1.1. Reduction of the problem to an integral equation

An unbounded elastic medium has a semi-infinite conical crack whose surface in a spherical system of coordinates (r, θ, φ) is described by the relations

$$R \le r < \infty, \quad -\pi \le \varphi \le \pi, \quad \theta = \omega \tag{1.1}$$

The centre of rotation is situated at a distance b from the origin of coordinates (see Fig. 1). The spherical torsion wave $u_\varphi^0(r, \theta, t)$ generated by it[12] at the instant of time $t=0$ touches the edge of the crack, the sides of which are assumed to be stress free. It is required to determine the dynamic stress intensity factor close to the crack for zero initial conditions.

In the above formulation, the only non-zero components of the displacement vector are the displacements $u_\varphi^0(r, \theta, t)$, which satisfy the torsion equation

$$(r^2 u_\varphi')' + \frac{(\sin\theta u_\varphi^\bullet)^\bullet}{\sin\theta} - \frac{u_\varphi}{\sin^2\theta} = \frac{r^2}{c^2}\frac{\partial^2 u_\varphi}{\partial t^2}, \quad c^2 = \frac{G}{\rho}, \quad \theta \ne \omega \tag{1.2}$$

everywhere with the exception of the surface of the crack (1.1). Here, G is the shear modulus, ρ is the density of the medium, a dot (•) denotes a partial derivative with respect to the variable θ, and a prime (′) denotes a partial derivative with respect to the variable r. On

☆ *Prikl. Mat. Mekh. Vol.* 74, No. 2, pp. 336–345, 2010.
* Corresponding author.
E-mail address: popov@onu.edu.ua (G.Ya. Popov).

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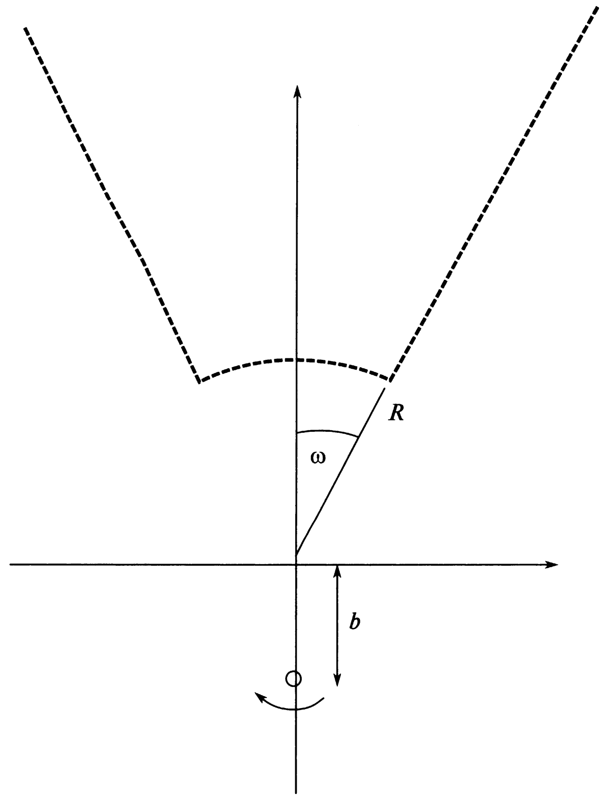

Fig. 1.

passing through the crack, the displacements and stresses undergo discontinuities of the first kind (jump discontinuities):

$$\langle u_\varphi(r,\omega,t)\rangle = u_\varphi(r,\omega-0,t) - u_\varphi(r,\omega+0,t)$$
$$\langle \tau_{\theta\varphi}(r,\omega,t)\rangle = \tau_{\theta\varphi}(r,\omega-0,t) - \tau_{\theta\varphi}(r,\omega+0,t) \quad (1.3)$$

The sides of the crack are stress-free:

$$\tau_{\theta\varphi}(r,\omega\pm 0,t) = 0 \quad (1.4)$$

The solution of the problem is constructed in the form of the superposition of the displacement and stress fields

$$u_\varphi(r,\theta,t) = u^0(r,\theta,t) + u^1(r,\theta,t), \quad \tau_{\theta\varphi}(r,\theta,t) = \tau^0(r,\theta,t) + \tau^1(r,\theta,t) \quad (1.5)$$

where u^0 and τ^0 are the displacements and stresses generated by the centre of rotation when there is no crack in the medium

$$u_\varphi^0(r,\theta,t) = \frac{\sin^2(f(\theta))}{\sin\theta}\frac{\partial}{\partial r}\left(r^{-1}H\left(t - \frac{\sin\theta(r-R)}{\sin(f(\theta))c}\right)\right), \quad f(\theta) = \operatorname{arctg}(\operatorname{cosec}\theta\cos\theta)b$$

($H(z)$ is Heaviside's function), and u^1 and τ^1 are the displacements and stresses resulting from the presence of the crack (1.1). The displacement $u^1(r,\ \theta,\ t)$ satisfies torsion equation (1.2).

We will change to dimensionless coordinates, for which we will introduce a change of variables

$$\rho = \frac{r}{R}, \quad \tau = \frac{ct}{R}, \quad u^1\left(\rho R,\theta,\frac{R\tau}{c}\right) \equiv w(\rho,\theta,\tau), \quad \tau^1\left(\rho R,\theta,\frac{R\tau}{c}\right) \equiv \tau(\rho,\theta,\tau) \quad (1.6)$$

The application of the integral Laplace transformation with respect to time

$$w_p(\rho,\theta) = \int_0^\infty e^{-p\tau} w(\rho,\theta,\tau)d\tau$$

leads, in Laplace-transform space, to the boundary-value problem

$$(\rho^2 w_p')' + \frac{(\sin\theta w_p^\bullet)^\bullet}{\sin\theta} - \frac{w_p}{\sin^2\theta} = \rho^2 p^2 w_p, \quad \theta \neq \omega; \quad \tau_p(\rho,\theta) = 0, \quad \theta = \omega \pm 0 \quad (1.7)$$

The shear stresses are connected with the displacement by the relation

$$\tau_p(\rho,\theta) = \frac{G}{R\rho}\left(w_p^{\bullet}(\rho,\theta) - \operatorname{ctg}\theta\, w_p(\rho,\theta)\right) \tag{1.8}$$

We will construct a discontinuous solution[13] of Eq. (1.7) by successively applying to it the integral Kontorovich–Lebedev transformations

$$w_{p\xi}(\theta) = \int_0^\infty \frac{K_{i\xi}(\rho)}{\sqrt{\rho}} w_p(\rho,\theta)\,d\rho$$

where $K_{i\xi}(z)$ is a MacDonald function of imaginary-order. Then, using a generalized scheme,[13] we apply the Legendre transformation

$$w_{p\xi n} = \left(\int_0^{\omega-0} + \int_{\omega+0}^{\pi}\right) P_n^1(\cos\theta)\sin\theta\, w_{p\xi}(\theta)\,d\theta, \quad w_{p\xi}(\theta) = \sum_{n=1}^{\infty} \frac{2n+1}{2n(n+1)} P_n^1(\cos\theta)\, w_{p\xi n}$$

where $P_n^1(\cos\theta)$ is the Legendre function of the first kind. Calculations using the generalized scheme of the method of integral transformations leads to a solution of Eq. (1.7) that satisfies this equation everywhere in the region $0 \le \rho \le \infty$, $0 \le \theta \le 2\pi$, with the exception of the surface of the crack (1.1) (on the crack, the values of the displacement jump $\langle w_p(\rho,\ \omega)\rangle$ and its normal derivative $\langle w_p^{\bullet}(\rho,\ \omega)\rangle$ are assumed to be those specified):

$$w_p(\rho,\theta) = \int_0^\infty \left\langle w_p^{\bullet}(\tilde\rho,\omega)\right\rangle Z\,d\tilde\rho - \int_0^\infty \left\langle w_p(\tilde\rho,\omega)\right\rangle \frac{\partial Z}{\partial\omega}\,d\tilde\rho \tag{1.9}$$

$$Z = Z(p\rho, p\tilde\rho;\theta,\omega) = \sin\omega \sum_{n=1}^{\infty} \frac{2n+1}{2n(n+1)} P_n^1(\cos\theta) P_n^1(\cos\omega) J_n(p\rho, p\tilde\rho)$$

$$J_n(p\rho, p\tilde\rho) = \frac{1}{p\sqrt{\rho\tilde\rho}} \begin{cases} I_{n+1/2}(p\rho) K_{n+1/2}(p\tilde\rho), & 0 < \rho < \tilde\rho \\ I_{n+1/2}(p\tilde\rho) K_{n+1/2}(p\rho), & 0 < \tilde\rho < \rho \end{cases}$$

In order to satisfy the boundary condition in problem (1.7), we will rewrite it, taking into account the change of variables (1.6) and the representation of the solution in form (1.5)

$$\begin{aligned} &G\left(w_p^{\bullet}(\rho,\omega-0) - \operatorname{ctg}\omega\, w_p(\rho,\omega-0)\right) + R\rho\tau_p^0(\rho,\omega) = 0 \\ &G\left(w_p^{\bullet}(\rho,\omega+0) - \operatorname{ctg}\omega\, w_p(\rho,\omega+0)\right) + R\rho\tau_p^0(\rho,\omega) = 0 \end{aligned} \tag{1.10}$$

Subtracting the second equation from the first, we obtain the relation

$$\left\langle w_p^{\bullet}(\rho,\omega)\right\rangle = \operatorname{ctg}\omega \left\langle w_p(\rho,\omega)\right\rangle \tag{1.11}$$

which relates the jump of the unknown function to the jump of its derivative.

To change to limit values of the discontinuous solution (1.9) and its normal derivative, we must investigate the asymptotic forms of the kernel $Z(p\rho,\ p\tilde\rho;\ \theta,\ \omega)$ as $n \to \infty$, and then repeat the procedure described earlier.[14] Isolating the logarithmic kernel of the function $Z(p\rho,\ p\tilde\rho;\ \theta,\ \omega)$, it can be shown that

$$w_p(\rho,\omega\pm 0) = \mp\frac{1}{2}\left\langle w_p(\rho,\omega)\right\rangle + \int_1^\infty \left\{ \left\langle w_p^{\bullet}(\tilde\rho,\omega)\right\rangle Z - \left\langle w_p(\tilde\rho,\omega)\right\rangle \frac{\partial Z}{\partial\omega} \right\}\Bigg|_{\theta=\omega} d\tilde\rho$$

$$w_p^{\bullet}(\rho,\omega\pm 0) = \mp\frac{1}{2}\left\langle w_p^{\bullet}(\rho,\omega)\right\rangle + \int_1^\infty \left\{ \left\langle w_p^{\bullet}(\tilde\rho,\omega)\right\rangle \frac{\partial Z}{\partial\theta} - \left\langle w_p(\tilde\rho,\omega)\right\rangle \frac{\partial Z}{\partial\theta\partial\omega} \right\}\Bigg|_{\theta=\omega} d\tilde\rho$$

(Note that the limits of integration with respect to the variable $\tilde\rho$ allow for non-zero jumps of the displacement and its normal derivative when $\tilde\rho > 1$.)

We will now realize one of the conditions (1.10), namely for $\theta = \omega - 0$, eliminating one of the jumps using (1.11). We obtain the integral equation

$$\int_1^\infty \chi_p(\tilde\rho) \left[2\frac{\partial Z}{\partial\theta} - \operatorname{tg}\omega \frac{\partial^2 Z}{\partial\theta\partial\omega} - \operatorname{ctg}\omega\ Z \right]_{\theta=\omega-0} d\tilde\rho = -\frac{R\rho}{G}\operatorname{tg}\omega\ \tau_p^0(\rho,\omega) \tag{1.12}$$

where $\chi_p(\tilde\rho) = \left\langle w_p(\tilde\rho,\ \omega)\right\rangle$ is the unknown jump of the displacement.

2. Solution of the integral equation

We will investigate the asymptotic forms of all three terms of the kernel of equation (1.12), using well-known formulae (Ref. 15, formula (7.13.2)). We will temporarily assume that the Laplace transform parameter p is a positive number; subsequently, we will apply an analytic continuation of its values. A Check shows that each integrand contains a singularity, and here the term with a second derivative has the strongest of these – a hypersingularity. The series recurring in these terms diverge, owing to the fact that, under the integration sign, we have introduced a differential operator

$$D_\rho = \rho^2 \frac{d^2}{d\rho^2} + 2\rho \frac{d}{d\rho} + \rho^2 p^2$$

This can be shown if we consider the relation proved earlier[14]

$$D_\rho J_n(p\rho, p\tilde{\rho}) = n(n+1) J_n(p\rho, p\tilde{\rho})$$

In order to leave only converging series in the kernel of the equation, we will take this differential operator outside the integration sign in Eq. (1.12). We transform integral equation (1.12) into the integral differential equation (IDE)

$$D_\rho \int_1^\infty \chi_p(\tilde{\rho}) \left[\sum_{i=1}^{3} S_i(p\rho, p\tilde{\rho}; \omega, \omega) \right] d\tilde{\rho} = -\frac{R\rho}{G} \tau_p^0(\rho, \omega) \tag{2.1}$$

$$S_1 = p \sin\omega \sum_{n=1}^{\infty} \frac{2n+1}{n^2(n+1)^2} \left[\frac{\partial}{\partial\omega} P_n^1(\cos\omega) \right]^2 J_n(p\rho, p\tilde{\rho})$$

$$S_2 = 2p \operatorname{ctg}\omega \sin\omega \sum_{n=1}^{\infty} \frac{2n+1}{n^2(n+1)^2} \left[\frac{\partial}{\partial\omega} P_n^1(\cos\omega) \right] P_n^1(\cos\omega) J_n(p\rho, p\tilde{\rho})$$

$$S_3 = p \operatorname{ctg}^2\omega \sin\omega \sum_{n=1}^{\infty} \frac{2n+1}{n^2(n+1)^2} \left[P_n^1(\cos\omega) \right]^2 J_n(p\rho, p\tilde{\rho})$$

We will transform the series in the expression for S_1 by using the asymptotic behaviour of the Bessel and Legendre functions of higher orders (Ref. 15, formula (3.9.1(1))) and introducing the function

$$h(\rho, \tilde{\rho}) = \begin{cases} (\rho/\tilde{\rho})^n, & \rho < \tilde{\rho} \\ (\tilde{\rho}/\rho)^{n+1}, & \tilde{\rho} < \rho \end{cases}$$

We will represent the series S_1 in the form of the sum of two series:

$$S_1 = \sum_{n=1}^{N} S_{1n} + \sum_{n=N+1}^{\infty} \tilde{S}_{1n}$$

where

$$S_{1n} = \sin\omega \frac{2n+1}{n^2(n+1)^2} \left[\frac{\partial}{\partial\omega} P_n^1(\cos\omega) \right]^2 J_n(p\rho, p\tilde{\rho})$$

and $\tilde{S}_{1n}$ is an asymptotically general term of the series

$$\tilde{S}_{1n} = \frac{1 + \sin(2n+1)\omega}{n\tilde{\rho}(n+1)} h(\rho, \tilde{\rho})$$

(N is a fairly large natural number). We will add and subtract in the representation of S_1 the sum $\sum_{n=1}^{N} \tilde{S}_{1n}$, after which, we will adopt, as the expression for S_1.

$$S_1 = \sum_{n=1}^{\infty} \tilde{S}_{1n} + \sum_{n=1}^{N} S_{1n} - \sum_{n=1}^{N} \tilde{S}_{1n}$$

Summing all the series in this expression,[16] in Laplace transform space we obtain the IDE for a new unknown function, related to the Laplace transform of the jump of displacements by the equality $\varphi_p(\tilde{\rho}) = \chi_p(\tilde{\rho})/\tilde{\rho}$:

$$D_\rho \int_1^\infty [\ln\frac{1}{|\rho-\tilde{\rho}|} + \alpha(\rho,\tilde{\rho}) + \beta(\rho,\tilde{\rho}) + \sum_{n=1}^N (S_{1n} - \tilde{S}_{1n})$$
$$+(S_2(p\rho,p\tilde{\rho}) + S_3(p\rho,p\tilde{\rho}))]\varphi(\tilde{\rho})d\tilde{\rho} = f_p(\rho,\omega)$$
$$\alpha(\rho,\tilde{\rho}) = \begin{cases} \dfrac{\rho-\tilde{\rho}}{\rho} + \dfrac{\tilde{\rho}}{\rho}\ln\rho, & \rho < \tilde{\rho} \\ \ln\rho, & \tilde{\rho} < \rho \end{cases} \tag{2.2}$$

$$\beta(\rho,\tilde{\rho}) = \left(\frac{\tilde{\rho}}{\rho}\right)^m \left[\frac{1}{2}\sin\omega\ln\left(1 - 2\frac{\rho}{\tilde{\rho}}\cos 2\omega + \left(\frac{\rho}{\tilde{\rho}}\right)^2\right) + \cos\omega\left(\frac{\tilde{\rho}}{\rho}\right)^m \operatorname{arctg}\left(\frac{\rho}{\tilde{\rho}}\sin 2\omega\left(1 - \frac{\rho}{\tilde{\rho}}\cos 2\omega\right)^{-1}\right)\right]$$

$$m = \begin{cases} 0, & \tilde{\rho} < \rho \\ 1, & \tilde{\rho} > \rho \end{cases}, \quad f_p(\rho,\omega) = -\frac{R\rho}{G}\tau_p^0(\rho,\omega)$$

The expressions for $S_2(p\rho,\ p\tilde{\rho})$ and $S_3(p\rho,\ p\tilde{\rho})$ are given above. IDE (2.2) is solved approximately using the method of orthogonal polynomials and time discretization.

In order to use the eigenvalue relation[17]

$$\left(1 - \frac{d^2}{d\rho^2}\right)\frac{1}{\pi}\int_1^\infty K_0(|\rho-\tilde{\rho}|)e^{-(\tilde{\rho}-1)}\sqrt{\tilde{\rho}-1}L_n^{1/2}(2\tilde{\rho}-2)d\tilde{\rho} = \frac{\sqrt{2}\Gamma(n+3/2)L_n^{1/2}(2\rho-2)}{n!e^{\rho-1}} \tag{2.3}$$

($L_n^{1/2}(z)$ are Chebyshev–Laguerre polynomials), we will present the logarithm using the MacDonald function (Ref. 18, formula (7.2.5(38)))

$$\ln\frac{1}{z} = K_0(z) - k(z), \quad k(z) = I_0(z)\ln 2 + (I_0(z) - 1)\ln\frac{1}{z} + \sum_{l=0}^\infty \frac{\varphi(l+1)}{(l!)^2}\left(\frac{z}{2}\right)^{2l} \tag{2.4}$$

The form of the function $\varphi(z)$ is given by a well-known formula (Ref. 18, formula (7.2.5(38))).

We transform the differential operator D_ρ by taking the factor ρ^2 outside the brackets, and in the brackets we add and substract unity

$$D_\rho = -\rho^2\left(1 - \frac{d^2}{d\rho^2} - \frac{2}{\rho}\frac{d}{d\rho} - p^2 - 1\right) = -\rho^2\left(1 - \frac{d^2}{d\rho^2}\right) - \rho^2\left(\frac{2}{\rho}\frac{d}{d\rho} + p^2 + 1\right) \tag{2.5}$$

Taking into account the representations (2.4) and (2.5), after dividing both parts of the equation by $-\rho^2$, we rewrite Eq. (2.2) in the form

$$\left(1 - \frac{d^2}{d\rho^2}\right)\int_1^\infty K_0(|\rho-\tilde{\rho}|)\varphi_p(\tilde{\rho})d\tilde{\rho} - \left(1 - \frac{d^2}{d\rho^2}\right)\int_1^\infty k(|\rho-\tilde{\rho}|)\varphi_p(\tilde{\rho})d\tilde{\rho}$$
$$-\left(1 - \frac{d^2}{d\rho^2}\right)\int_1^\infty\left[\alpha(\rho,\tilde{\rho}) + \beta(\rho,\tilde{\rho}) - \sum_{n=1}^N \tilde{S}_{1n}\right]\varphi_p(\tilde{\rho})d\tilde{\rho}$$
$$-\left(\frac{2}{\rho}\frac{d}{d\rho} + p^2 + 1\right)\int_1^\infty \tilde{\rho}\sum_{n=1}^\infty \frac{(2n+1)\sin\omega}{n^2(n+1)^2}\left[\frac{\partial P_n^1(\cos\omega)}{\partial\omega}\right]^2 J_n(p\rho,\tilde{\rho}p)]\varphi_p(\tilde{\rho})d\tilde{\rho}$$
$$-\left(1 - \frac{d^2}{d\rho^2}\right)\int_1^\infty\sum_{n=1}^N S_{1n}\varphi_p(\tilde{\rho})d\tilde{\rho} - D_\rho\left[\int_1^\infty\sum_{n=2}^3 S_i(p\rho,p\tilde{\rho})\varphi_p(\tilde{\rho})\right]d\tilde{\rho} = -\frac{f_p(p,\omega)}{\rho^2} \tag{2.6}$$

We apply to Eq. (2.6) the inverse Laplace transform, taking well-known relations into account.[19] We obtain

$$L^{-1}\left[-\frac{f_p(\rho,\omega)}{\rho^2}\right] = \frac{R}{G\rho}\tau_0(\rho,\omega,\tau), \quad L^{-1}[\varphi_p(\tilde{\rho})] = L^{-1}\left[\frac{\chi_p(\tilde{\rho})}{\tilde{\rho}}\right] = \frac{\chi(\tilde{\rho},\tau)}{\tilde{\rho}}$$
$$L^{-1}[J_n(p\rho,p\tilde{\rho})] = q(\tau), \quad L^{-1}[pJ_n(p\rho,p\tilde{\rho})] = \frac{dq(\tau)}{d\tau}, \quad L^{-1}\left[p^2 S_n(p\rho,p\tilde{\rho})\right] = \frac{d^2q(\tau)}{d\tau^2} \tag{2.7}$$

The function $q(\tau)$ is defined by a tabulated inversion formula.[19]

Further application of the convolution theorem for the Laplace transform enables us to obtain the IDE in the original space

$$\left(1-\frac{d^2}{d\rho^2}\right)\int_1^\infty K_0(|\rho-\tilde\rho|)\frac{\chi(\tilde\rho,\tau)}{\tilde\rho}d\tilde\rho-\left(1-\frac{d^2}{d\rho^2}\right)\int_1^\infty l(\rho,\tilde\rho)\frac{\chi(\tilde\rho,\tau)}{\tilde\rho}d\tilde\rho$$
$$-\int_0^\tau\int_1^\infty \gamma(\rho,\tilde\rho,\tau-t)\frac{\chi(\tilde\rho,t)}{\tilde\rho}d\tilde\rho dt=\frac{R}{G\rho}\tau_0(\rho,\tau) \tag{2.8}$$

$$l(\rho,\tilde\rho)=k(|\rho-\tilde\rho|)+\alpha(\rho,\tilde\rho)+\beta(\rho,\tilde\rho)-\sum_{n=1}^N\tilde S_{1n}$$

$$\gamma(\rho,\tilde\rho,\tau)=L^{-1}\left[\left(\frac{2}{\rho}\frac{d}{d\rho}+p^2+1\right)\sum_{n=1}^\infty\frac{(2n+1)\sin\omega}{n^2(n+1)^2}\left[\frac{\partial P_n^1(\cos\omega)}{\partial\omega}\right]^2 J_n(p\rho,p\tilde\rho)\right]$$
$$+L^{-1}\left[\left(1-\frac{d^2}{d\rho^2}\right)\sum_{n=1}^N S_{1n}\right]+L^{-1}\left[D_\rho\sum_{i=2}^3 S_i(p\rho,p\tilde\rho)\right]$$

Laplace inversion is carried out by means of formulae (2.7).

The perturbed field is investigated in the time interval $\tau\in[0,\ T]$, which is divided into segments $[\tau_k,\ \ \tau_{k+1}]$, $\tau_k=kT/M$ ($k=1, 2, \ldots, M$) with a step $h=T/M$. The integral over the variable τ in Eq. (2.8) is replaced by Simpson's quadrature formula, the nodes of which are denoted by A_k. We have

$$\left(1-\frac{d^2}{d\rho^2}\right)\frac{1}{\pi}\int_1^\infty K_0(|\rho-\tilde\rho|)\chi_k(\tilde\rho)d\tilde\rho-\left(1-\frac{d^2}{d\rho^2}\right)\frac{1}{\pi}\int_1^\infty k(|\rho-\tilde\rho|)\chi_k(\tilde\rho)d\tilde\rho$$
$$-\sum_{k=1}^M A_k\int_1^\infty P(\rho,\tilde\rho,\tau_M-\tau_k)\chi_k(\tilde\rho)d\tilde\rho=\frac{R}{G\rho}\tau_0(\rho,\tau_k),\quad k=1,2,\ldots,M;\quad \tilde\chi_k(\tilde\rho)=\frac{\chi(\tilde\rho,\tau_k)}{\tilde\rho} \tag{2.9}$$

On the basis of the presence of eigenvalue relation (2.3), and the fact that the displacement jump on the crack is an unknown function, we search for the solution of Eq. (2.9) in the form of the series

$$\chi_k(\tilde\rho)\equiv\chi(\tilde\rho,\tau_k)=\sum_{l=0}^\infty\sqrt{\tilde\rho-1}e^{-(\tilde\rho-1)}L_l^{1/2}(2\tilde\rho-2)\chi_l^{(k)} \tag{2.10}$$

We substitute expansion (2.10) into Eq. (2.9) and realize the standard scheme of the orthogonal polynomial method. As a result, we obtain a system of infinite linear algebraic equations

$$\chi_l^{(m)}+\sum_{l=0}^\infty k_{lm}\chi_l^{(m)}+\sum_{k=1}^M A_k\sum_{l=0}^\infty P_{klm}\chi_l^{(k)}=F_l^{(m)},\quad m=1,2,\ldots,M,\quad l=0,1,2,\ldots \tag{2.11}$$

with coefficients

$$k_{lm}=C_m\int_0^\infty\int_0^\infty k(|\rho-\tilde\rho|)\sqrt{\rho}\sqrt{\tilde\rho}e^{-(\tilde\rho-1)}e^{-\rho}L_l^{1/2}(2\tilde\rho-2)L_m^{1/2}(2\rho-2)d\tilde\rho d\rho$$
$$P_{klm}=\int_0^\infty\int_0^\infty P(\rho,\tilde\rho,\tau_m-\tau_k)\sqrt{\tilde\rho}\sqrt{\rho}e^{-\tilde\rho}e^{-\rho}L_l^{1/2}(2\tilde\rho-2)\frac{L_m^{1/2}(2\rho-2)}{C_m}d\tilde\rho d\rho$$
$$F_l^{(m)}=\frac{A\sin^2\omega}{G\sqrt{\cos\omega}}\frac{\tau_m^{3/2}L_l^{1/2}(2\tau_l/\cos\omega)}{(\tau_m+\cos\omega)^2e^{\tau_m/\cos\omega}C_l},\quad C_l=\left(\frac{\Gamma(l+3/2)}{l!}\right)^2$$

System (2.11) is initially solved for the case $M=1$. This leads to the solution of the infinite system for the coefficients of expansion $\chi_l^{(1)}$ ($l=0, 1, 2, \ldots$) by the reduction method (its applicability is proved by a scheme proposed earlier[13]). For subsequent values of m, the system is solved recurrently by transferring unknown terms to the right-hand side

$$\chi_l^{(m)}+\sum_{l=0}^\infty k_{lm}\chi_l^{(m)}+A_m\sum_{l=0}^\infty P_{lm}\chi_l^{(m)}=F_l^{(m)}-\sum_{i=1}^{m-1}A_i\sum_{l=0}^\infty P_{ilm}\chi_l^{(i)},\quad m=2,3,\ldots,M \tag{2.12}$$

Having obtained a solution of the system of the form (2.12), we thereby determine values of the unknown jump of displacements at the k-th instant of time ($k=1, 2, \ldots, M$) by means of formula (2.10).

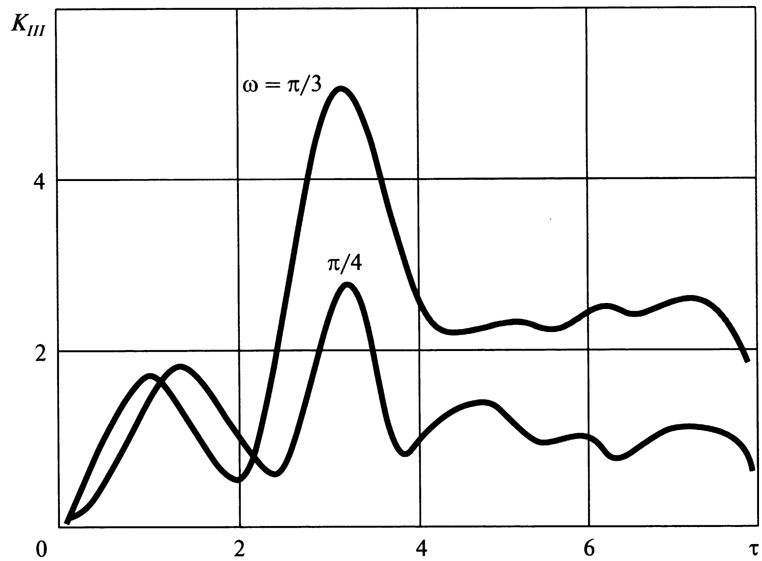

Fig. 2.

3. Determination of the dynamic

3.1. Stress intensity factor (SIF)

The SIF values are defined by the formula

$$K_{III}(\tau) = \lim_{r\to R-0} \sqrt{2\pi(R-r)}\tau_\theta(r,\omega,\tau) = \lim_{\rho\to 1-0} \sqrt{2\pi R}\sqrt{1-\rho}\tau_\theta(\rho,\omega,\tau) \tag{3.1}$$

The stresses are determined using formula (1.5), where $\tau^0(r, \omega, \tau)$ is a continuous term not contributing to the calculation of the SIF; the values of the stresses $\tau^1(r, \omega, \tau)$ are defined by the left-hand side of Eq. (2.9), namely by the first term, which contains the highest-order singularity. Thus, the SIF at the k-th instant of time is defined by the expression

$$K_{III}(\tau_k) = K_{III}^{(k)} = \lim_{\rho\to 1-0} \sqrt{2\pi R}\sqrt{1-\rho}\left(1-\frac{d^2}{d\rho^2}\right)\frac{1}{\pi}\int_1^\infty K_0(|\rho-\tilde{\rho}|)\chi_k(\tilde{\rho}), \quad k = 1,2,\ldots,M \tag{3.2}$$

The functions $\chi_k(\rho)$ are defined by expansion (2.10). To calculate the SIF, it is necessary to use analytic continuation of eigenvalue relation (2.3). It has been shown[17] that the following relation exists

$$B_j(\zeta) = \left(1-\frac{d^2}{d\xi^2}\right)\frac{1}{\pi}\int_0^\infty \sqrt{\sigma}e^{-\sigma}L_j^{1/2}(2\sigma)K_0(|\xi-\sigma|)d\sigma = \frac{\Gamma(j+3/2)}{\pi j!}I^*(\xi), \quad \xi < 0 \tag{3.3}$$

$$I^*(\xi) = \int_0^\infty \frac{t^{j+\frac{1}{2}}e^{t\xi}}{(t+2)^{j+1}}dt, \quad \lim_{\xi\to -0}\sqrt{\xi}B_j(\xi) = \frac{\Gamma(j+3/2)}{(-1)^{j+2}\sqrt{2\pi}j!}$$

We substitute expansion (2.10) under the integration sign in formula (3.2) and make the replacement of variables $\xi = \rho - 1$. Taking formulae (3.3) into account, we finally obtain

$$K_{III}^{(k)} = \frac{1}{\pi}\sum_{l=0}^\infty \frac{\Gamma(l+3/2)}{(-1)^{l+2}l!}\chi_l^{(k)}$$

The coefficients $\chi_l^{(k)}$ are determined from system (2.11).

4. Discussion of numerical results

As the elastic medium, we examined steel with crack opening angles $\omega = \pi/3$ and $\omega = \pi/4$ (Fig. 2) and with the centre of rotation positioned as shown in Fig. 1. It can be seen that an increase in the crack opening angle leads to an increase in SIF values, and, from the sixth instant of dimensionless time onwards, there is a tendency for a steady regime of vibrations to be established. The calculation algorithm enables us to obtain SIF values for up to 12–14 instants of dimensionless time; then, in view of the build-up of errors on the right-hand side of systems of the form (2.12), the calculation becomes unstable.

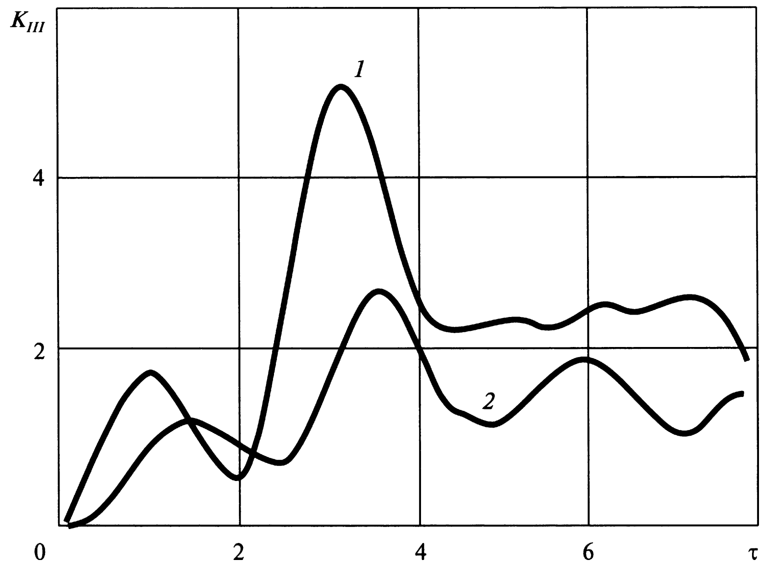

Fig. 3.

The SIF values were also investigated as a function of the position of the centre of rotation. Curve *1* in Fig. 3 for a crack opening angle $\omega = \pi/3$ corresponds to the position of the centre of rotation indicated in Fig. 1 (curve *2*) – the position of the centre of rotation at a distance b ($b > R$) from the origin of coordinates in the positive direction of OY axis. It can be seen that the SIF is considerably less in the latter case, which is due to the peculiar screening effect created by the crack.

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